



Narain transform for spectral deformations of random matrix models

M.A. Nowak^{*}, W. Tarnowski

*Institute of Theoretical Physics and Mark Kac Complex Systems Research Center, Jagiellonian University,
S. Łojasiewicza 11, PL 30-348 Kraków, Poland*

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Abstract

We start from applying the general idea of spectral projection (suggested by Olshanski and Borodin and advocated by Tao) to the complex Wishart model. Combining the ideas of spectral projection with the insights from quantum mechanics, we derive in an effortless way all spectral properties of the complex Wishart model: first, the Marchenko-Pastur distribution interpreted as a Bohr-Sommerfeld quantization condition for the hydrogen atom; second, hard (Bessel), soft (Airy) and bulk (sine) microscopic kernels from properly rescaled radial Schrödinger equation for the hydrogen atom. Then, generalizing the ideas based on Schrödinger equation to the case when Hamiltonian is non-Hermitian, we propose an analogous construction for spectral projections of universal kernels for bi-orthogonal ensembles. In particular, we demonstrate that the Narain transform is a natural extension of the Hankel transform for the products of Wishart matrices, yielding an explicit form of the universal kernel at the hard edge. We also show how the change of variables of the *rescaled* kernel allows us to make the link to the universal kernel of the Muttalib-Borodin ensemble. The proposed construction offers a simple alternative to standard methods of derivation of microscopic kernels. Finally, we speculate, that a suitable extension of the Bochner theorem for Sturm-Liouville operators may provide an additional insight into the classification of microscopic universality classes in random matrix theory.

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^{*} Corresponding author.

E-mail addresses: maciej.a.nowak@uj.edu.pl (M.A. Nowak), wojciech.tarnowski@doctoral.uj.edu.pl (W. Tarnowski).

1. Introduction

Determinantal point processes [1] appear in several areas of mathematics, physics and applied sciences, ranging from random matrix theory (RMT) to combinatorics and theory of representations. The unique feature of such processes relies on the fact, that the N -point joint probability distribution function is expressed as a determinant of a matrix built from a single, two-point correlation function known as a kernel. Celebrated examples of such kernels in high energy physics include “hard-edge” Bessel kernels [2–4], observed in numerous lattice calculations [5], or Pearcey kernels appearing at strong-weak coupling phase transition in Yang-Mills theories in the limit of large number of colors [6,7]. The calculation of kernels and their asymptotic limits became therefore an area of vigorous studies using advanced mathematical tools, like supersymmetry [8–11], orthogonal [12] and bi-orthogonal polynomials [13], Riemann-Hilbert problem [14–16] and Plancherel-Rotach [17] limiting procedures for integral representations, to mention most popular.

Borodin and Olshanski [18] offered a different point of view at kernels built from orthogonal polynomials in random matrix theory. When treated as an integral operator, the kernel is a projection – a consequence of a finite number of eigenvalues and orthogonality of polynomials. This idea was later advocated by Tao [19], who also used physical intuition by the mapping between Gaussian Unitary Ensemble and the quantum harmonic oscillator. In this quantum mechanical picture the projection stems from the fact that the first N energy levels are occupied. Using these techniques, Bornemann elaborated the Sturm-Liouville problem and showed that all three classical limiting kernels can be obtained in this way [20]. The joint probability density functions in RMT can be mapped onto various physical systems, including point charges with logarithmic interaction [21–23] and spinless fermions [24]. The latter equivalence proved to be a fertile ground of applications of RMT to cold atom systems [25–28].

It is intriguing to investigate the chronological intertwining of the ideas in quantum mechanics, mathematics and statistics from the perspective of the contemporary random matrix theory. In 1926, Schrödinger has solved his equation for Coulomb potential, obtaining among others the radial part of the wave function in terms of Laguerre functions.¹ Two years later (1928) Wishart introduced his ensemble in multivariate statistics, as a generalization of the χ^2 ensemble [29]. The original paper deals with the real random variables, but his ideas were later generalized to complex variables [30]. A year later (1929), Bochner has proven his theorem [31] (see section 3.1) for Sturm-Liouville operators, without any direct references to Schrödinger equation. At that time spectral properties of random matrices were not considered at all. Laguerre polynomials appeared explicitly in random matrix theory only after Mehta and Gaudin used the orthogonal polynomial method to disentangle the Van der Monde determinant [32]. This technique has also paved the way for classical universal kernels. However, the link to the uniqueness of the determinantal triality of soft, edge and bulk microscopic universalities of Sturm-Liouville operators have been cleared out only recently [20].

With introducing non-trivial initial conditions for Dyson Brownian motion, new universality classes emerged in random matrix theory. In the 90’s of the previous century, collision of soft edges in GUE led Brezin and Hikami [33] to the Pearcey kernel. In a similar collision of chiral fronts at the hard edge of the chiral random matrix model one of the authors found the Bessoid kernel universality [34]. While still determinantal [35], such models break rotational invariance,

¹ Year earlier, Pauli has quantized algebraically hydrogen atom, using the hidden symmetry (Runge-Lenz vector) of the Coulomb potential, therefore treating this problem as a free problem on S_3 hypersphere.

and require non-standard tools. Later it was discovered that such ensembles can be solved by polynomials that are orthogonal with respect to more than one weight [36].

The bi-orthogonality method of Muttalib and Borodin [13,37] opened a new way for treating a broader class of random matrix models, to which the orthogonal polynomial method does not apply. Historically, it is again puzzling that bi-orthogonality was not linked to random matrices earlier. Already in 1951, Fano and Spencer [38] studying propagation of the X-rays through the matter, have introduced bi-orthogonal Laguerre polynomials. These ideas were further developed in mathematics by Preiser [39] and Konhauser [40]. In particular, Preiser's construction corresponds exactly to the case of Muttalib-Borodin ensemble.

The aim of this work is to further elaborate the spectral projection method, with the use of insights from elementary quantum mechanics. In section 2, we pedagogically introduce the spectral projection method and demonstrate its easiness in taking the microscopic limits by recalculating all limiting kernels in the complex Wishart ensemble. We link the Marchenko-Pastur distribution [41] to the Bohr-Sommerfeld quantization condition. We also notice that the threeness of the classical universal kernels can be linked to the strictures originating from the Bochner theorem for Sturm-Liouville problem [31].

Recent developments on the integrable structure of products of random matrices and the multitude of new microscopic kernels in biorthogonal ensembles naturally pose a question whether the spectral projection method can be extended to incorporate these universality classes. In section 3 we discuss the possibilities to circumvent the constraints of Bochner's theorem and consider an analog of a quantum-mechanical Hamiltonian, but with higher number of derivatives. Although such an operator may not be self-adjoint, still, due the fact that its left and right eigenvectors form a bi-orthogonal basis, it is possible to infer the microscopic limit of the kernels, using the spectral projection method. The power of this approach – easiness of calculation of microscopic kernel without the need of Plancherel-Rotach asymptotics – is demonstrated on two examples: singular values of products of Gaussian matrices [42] and the Muttalib-Borodin ensemble [13,37]. In both cases the Narain transform [43–45] allows one to recover the Meijer-G hard edge universality, generalizing Bessel kernel. Again, the spectral projection translates to the truncation of the phase space of the associated transform.

Section 4 concludes the paper. In appendix A, we show an alternative mapping of the Wishart ensemble to the 2-dimensional hydrogen atom problem [46]. In appendix B we recover the Marchenko-Pastur distribution from the WKB approximation. In appendix C we recall some properties of the Meijer-G functions.

2. Spectral projections from hydrogen atom problem

2.1. Complex Wishart ensemble

Let us consider Hermitian matrix $M = XX^\dagger$, where X is the complex $N \times T$ matrix with entries given by the probability density function $P(X)dX = Z_{NT}^{-1} e^{-\frac{1}{\sigma^2} \sum_{\alpha,j}^{N,T} |X_{\alpha,j}|^2} \prod_{\alpha,j}^{N,T} d\Re X_{\alpha,j} \times d\Im X_{\alpha,j}$. Here Z_{NT}^{-1} provides the normalization and σ^2 is the variance of the complex Gaussian distribution, which we set to 1, to simplify the expressions. This defines complex Wishart matrix [29]. Switching to eigenvalues, we arrive, using standard methods [47], at their joint probability density

$$P(\lambda_1, \dots, \lambda_N) = Q_N^{-1} \prod_{j=1}^N \lambda_j^\alpha e^{-\lambda_j} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2, \quad (1)$$

with $\alpha = T - N$, and the Vandermonde determinant (last term) is the price for switching from elements of X to eigenvalues λ_i of matrix M . Standard orthogonal polynomials method [12] allows one to rewrite the probability distribution as

$$P_N(\lambda_1, \dots, \lambda_N) = \frac{1}{N!} \left(\det [\psi_{j-1}(\lambda_k)]_{j,k=1}^N \right)^2 = \frac{1}{N!} [\det K_N(\lambda_i, \lambda_j)], \quad (2)$$

with the correlation *kernel*

$$K_N(\lambda, \mu) = \sum_{l=0}^{N-1} \psi_l(\lambda) \psi_l(\mu), \quad (3)$$

where $\psi_l(\lambda) = e^{-\lambda/2} \lambda^{\alpha/2} P_l(\lambda)$ and P_l are monic polynomials. This form already suggests links to quantum mechanics. The first equality in (2) represents the joint probability of eigenvalues as the square of the Slater determinant, therefore can be interpreted as the quantum probability density of non-interacting spinless fermions (see [28] for a review). This also explains why the eigenvalue density is expressed solely in terms of a two-point function (second expression on the r.h.s. of (2)). Next, we see that the most natural choice of polynomials is dictated by the weight $w_\alpha(\lambda) = \lambda^\alpha e^{-\lambda}$. Such polynomials, orthonormal on the positive part of the real axis, are the associated Laguerre polynomials and appear in the radial part of the Schrödinger equation. Indeed, upon standard separation of variables in the wavefunction, $\varphi(\vec{r}) = R(r)Y_l^m(\theta, \psi)$, it reads

$$\frac{d^2 y(r)}{dr^2} + \left[\frac{2\mu e^2}{r\hbar^2} - \frac{l(l+1)}{r^2} \right] y(r) = -\frac{2\mu E}{\hbar^2} y(r), \quad (4)$$

where $y(r) = rR(r)$. Switching to dimensionless variable $x = r\epsilon$, where $(\epsilon/2)^2 = -2\mu E/\hbar^2$, putting $2\mu = 1$ and all other physical constants to 1, we recover [48]

$$\frac{d^2 y(x)}{dx^2} + \left[-\frac{1}{4} + \frac{1}{\epsilon x} - \frac{l(l+1)}{x^2} \right] y(x) = 0, \quad (5)$$

where $y = y_n^l = e^{-x/2} x^{(k+1)/2} L_j^k(x)$. Here $k = 2l + 1$ and the principal quantum number is related to the order of Laguerre polynomial as $n = j + l + 1$. Note, that $\epsilon = 1/n$, or, equivalently, $E_n = -1/4n^2$, since in our units Bohr's radius equals to 2. To map this random matrix problem to the hydrogen atom we associate $\psi_l(\lambda) = \sqrt{x}y(x)$. This completes the dictionary between hydrogen atom problem and the Wishart kernel. In Appendix A we also present a mapping into 2D hydrogen atom with $1/r$ potential [46], in which the relation between eigenfunctions of the radial part of the Schrödinger equation and ψ is even more explicit.

The Schrödinger equation for ψ expressed in terms of the parameters of the Wishart ensemble reads

$$\frac{d^2 \psi_k}{dx^2} + \frac{1}{x} \frac{d\psi_k}{dx} + \frac{1+2k+\alpha}{2x} \psi_k - \frac{\alpha^2}{4x^2} \psi_k = \frac{1}{4} \psi_k. \quad (6)$$

Finally, let us note that in the bra-ket notations the kernel can be rewritten as $\hat{K}_N = \sum_{k=0}^{N-1} |\psi_k\rangle \langle \psi_k|$, thus it is the operator projecting onto the set of N lowest eigenstates. Indeed, due to the orthonormality of eigenfunctions $\hat{K}_N^2 = \hat{K}_N$, last equation, when calculated in coordinate representation, yields well-known reproducing property $\langle x | \hat{K}_N | y \rangle \equiv K_N(x, y) = \int K_N(x, z) K_N(z, y) dz$.

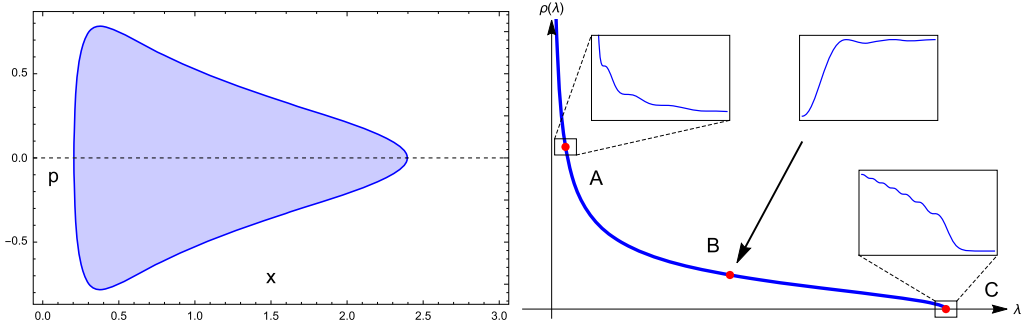


Fig. 1. (left) A region in the phase space where the Wigner function is nonzero. We took $c=0.3$. (right) Identification of three regions of the Marchenko-Pastur distribution that give rise to different microscopic scalings.

2.2. Macroscopic density from the semiclassical approximation

To have the finite support of the spectral density in the large N limit, we rescale $x \rightarrow Tx$. Upon this scaling and identifying momentum² as $p = -\frac{i}{T} \frac{d}{dx}$ (in analogy to $\hbar \leftrightarrow 1/T$) in the limit $N, T \rightarrow \infty$ with $c = N/T$ fixed we obtain the Schrödinger equation $(p^2 + V_{eff})\psi = -\frac{1}{4}\psi$ with the effective potential

$$V_{eff} = \frac{(1-c)^2}{4x^2} - \frac{1+c}{2x}. \quad (7)$$

The probability density function can be obtained from the Wigner quasiprobability distribution, defined as

$$W(x, p) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \bar{\psi}(x+y)\psi(x-y)e^{2ipy/\hbar} dy, \quad (8)$$

by integrating out the momentum. In the large N limit the Wigner function is constant in the region of the phase space $p^2 + V(x) \leq \frac{1}{4}$ and zero outside [49] (see Fig. 1). As a consequence, the density of eigenvalues is proportional to the momentum and the Bohr-Sommerfeld quantization condition

$$T \oint p(xT) dx = \left(N + \frac{1}{2}\right) 2\pi \quad (9)$$

on the RMT side corresponds to the normalization of the density

$$\int_{r_-}^{r_+} \rho(x) dx = 1. \quad (10)$$

This allows us to obtain the density of eigenvalues

$$\rho(x) = \frac{1}{2\pi cx} \sqrt{(r_+ - x)(x - r_-)}. \quad (11)$$

² For this analogy it is even better to take the 2D radial momentum $p_r = \frac{i}{T} \left(\frac{d}{dx} + \frac{1}{x} \right)$, but this eventually leads to the same result in large T limit.

Here $r_{\pm} = (1 \pm \sqrt{c})^2$ are classical turning points in WKB approximation. In Appendix B we provide another derivation of this result based on the explicit WKB analysis of (6).

We have therefore obtained Marchenko-Pastur distribution as an *exact, semiclassical limit of the quantum mechanical hydrogen atom problem*. It is intriguing to speculate why such a link has not been exploited (to the authors knowledge) in the literature. Perhaps the reason is that the Bohr-Sommerfeld quantization condition does not reproduce correctly the ground state of the hydrogen atom, and not even the Bohr quantization condition [50]. It is amusing to notice, that if one replaced $l(l+1)$ by $(l+1/2)^2$ in the numerator of the centrifugal potential, this would be the case and B-S approximation would lead to the exact result for the hydrogen spectra [51]. Of course, in the large l limit it does not matter which of the equations (5) or (6) we use, however, at the microscopic level, the additional square root in Laguerre function for the Wishart will play the crucial role in getting the proper scaling of the hard edge.

We complete this part with the observation, that in the case of harmonic oscillator, similar construction is ambiguities free, since Bohr-Sommerfeld quantization condition yields exact spectrum. The Wigner semicircle, or rather semi-ellipse, is just the similar projection of the ellipse $p^2 + x^2/4 = 1$ onto the x axis in the phase space. The Bohr-Sommerfeld quantization condition just reads $\int \rho(x) dx = 1$, where $\rho = \frac{1}{2\pi} \sqrt{4 - x^2}$ (in units where $2\mu = 1$) [19]. Again, the rigid argument comes from the fact, that the Wigner function for harmonic oscillator is explicitly known [49], and yields a direct relation between the momenta and positions at the semi-classical level.

2.3. Microscopic scaling as a spectral deformation

Correlations of eigenvalues probed on the scale of the typical separation between them are independent on the probability density function of matrix elements. They fall into several universality classes, depending on the point x_0 of the spectrum at which their behavior is probed. The shape of the spectral density, in turn, determines the microscopic scale s by demanding that in the interval $[x_0, x_0 + s]$ one expects one eigenvalue to occur. Looking at the form of the Marchenko-Pastur distribution (see Fig. 1), we immediately identify three distinct regions corresponding to microscopic scalings.

A Hard edge. In the limit when $N, T \rightarrow \infty$ but $\alpha = T - N$ remains fixed ($c \rightarrow 1$), the turning point r_- approaches zero, and the eigenvalue density near this point behaves like $1/\sqrt{x}$. Asking how many out of original N eigenvalues will appear in a narrow bin of size s around zero, we get

$$n_{hard} \sim N \int_0^s \frac{dx}{\sqrt{x}} \sim N\sqrt{s}. \quad (12)$$

Demanding that $n_{hard} \sim 1$, we set the proper microscopic scale to $s \sim N^{-2}$.

B Bulk. Between the endpoints, at some x_0 , when counting the number of eigenvalues in a narrow interval of length s , one can approximate the density as locally constant $\rho(x_0)$. This leads to

$$n_{bulk} \sim N \int_{x_0-s/2}^{x_0+s/2} \rho(x_0) dx \sim Ns\rho(x_0), \quad (13)$$

which implies that the bulk microscopic scale is $s \sim \frac{1}{N\rho(x_0)}$.

C Soft edge. When $c \neq 1$, the macroscopic spectral density around both turning points vanishes like $\sqrt{|r_{\pm} - x|}$. Counting the eigenvalues close to the edge, leads to

$$n_{soft} \sim N \int_0^s \sqrt{x} dx \sim N s^{3/2}, \quad (14)$$

thus the edge microscopic scale is set to $s \sim N^{-2/3}$.

Following the generic arguments by Borodin and Olshanski [52] and inspired by Tao [19] presentation for the Gaussian Unitary Ensemble, we will now obtain the microscopic, universal kernels for the complex Wishart ensemble. We remark that this case belongs to the generic class of Sturm-Liouville operators, considered recently by Bornemann [20]. However, in this note, we attempt to use the insights from quantum mechanics rather than abstract mathematics.

The complete set of eigenfunctions provides a resolution of identity $\mathbf{1} = \sum_{k=0}^{\infty} |\psi_k\rangle \langle \psi_k|$. The random matrix kernel is obtained by truncating this sum to first N eigenstates and is therefore a projection. Formally, we project the Hilbert space to the space of functions $|\psi\rangle$ that satisfy $\langle \psi | \hat{H} | \psi \rangle \leq E_{N-1}$. Using the explicit form of (6), we write is less formally as

$$\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + \frac{1+2k+\alpha}{2x} - \frac{\alpha^2}{4x^2} \geq \frac{1}{4} \quad (15)$$

and later we are looking for a convenient representation of functions that span such space.

The microscopic scalings provide further deformations of the projection range, which in the large N, T limit gives rise to the universal microscopic kernels, which we work out in details beneath.

A Bessel kernel. Using the hard edge scaling $x/T \rightarrow sN^{-2}$, and performing the large N limit (note that $k \sim N$), we obtain the equation

$$\frac{d^2}{ds^2} + \frac{1}{s} \frac{d}{ds} + \frac{1}{s} - \frac{\alpha^2}{4s^2} \geq 0. \quad (16)$$

Changing variables $z = 2\sqrt{s}$ converts the above bound into the more familiar form

$$\Delta_{\alpha} \equiv -\frac{d^2}{dz^2} - \frac{1}{z} \frac{d}{dz} + \frac{\alpha^2}{z^2} \leq 1, \quad (17)$$

where on the l.h.s. we recognize the Bessel operator, appearing in quantum mechanical problems with polar angle symmetry. To see the deformation caused by the microscopic scaling at the hard edge, we invoke the *Hankel transform*

$$F(t) = H_{\alpha}[f(z)] = \int_0^{\infty} J_{\alpha}(tz) f(z) z dz \quad (18)$$

and its inverse

$$f(z) = \int_0^{\infty} J_{\alpha}(tz) F(t) t dt. \quad (19)$$

The Hankel transform of the Bessel operator reads $H_\alpha[\Delta_\alpha f(z)] = t^2 F(t)$ [53], thus the spectral deformation in dual variable t (note that t cannot be negative) reads simply

$$t \leq 1. \quad (20)$$

Hankel transform and its inverse give a representation of the identity operator

$$f(z') = \int_0^\infty \int_0^\infty z t J_\alpha(z't) J_\alpha(tz) f(z) dz dt. \quad (21)$$

The deformation condition (20) restricts the range of the parameter t and therefore turns the above identity operator into the projection

$$\mathbf{P}[f(z')] = \int_0^\infty \left[\int_0^1 z t J_\alpha(z't) J_\alpha(tz) dt \right] f(z) dz. \quad (22)$$

Changing variables once more as $t = \sqrt{s}$ and introducing $z = \sqrt{y}$ and $z' = \sqrt{x}$, we rewrite the above as

$$\mathbf{P}[f(x)] = \int_0^\infty \left[\frac{1}{4} \int_0^1 J_\alpha(\sqrt{xs}) J_\alpha(\sqrt{ys}) ds \right] f(y) dy \equiv \int_0^\infty K(x, y) f(y) dy, \quad (23)$$

so the kernel, understood as a projection, reads

$$K_{Bessel}(x, y) = \frac{1}{4} \int_0^1 J_\alpha(\sqrt{xs}) J_\alpha(\sqrt{ys}) ds = \frac{J_\alpha(\sqrt{x}) J'_\alpha(\sqrt{y}) \sqrt{y} - \sqrt{x} J'_\alpha(\sqrt{x}) J_\alpha(\sqrt{y})}{2(x - y)}, \quad (24)$$

where on the r.h.s. we presented the more familiar form of the kernel based on the Lommel integral and primes denote differentiation with respect to the argument. Hard edge scaling deforms the upper half plane in s variable onto the strip between the parallel lines $s = 0$ and $s = 1$.

B Sine kernel. Combining the rescaling needed for the finite support and the microscopic scaling, we define the new variable s as $x/T = x_0 + \frac{s}{N\rho(x_0)}$. Upon taking the large N, T limit, the bound (15) in this new variable reads

$$\frac{d^2}{ds^2} \geq \frac{(x_0 - r_+)(x_0 - r_-)}{4c^2 x_0^2 \rho^2(x_0)}. \quad (25)$$

Using the explicit form of the Marchenko-Pastur density (11), the above bound is simplified to

$$-\frac{d^2}{ds^2} \leq \pi^2. \quad (26)$$

On the l.h.s. we recognize the Schrödinger operator for a free particle, therefore the natural procedure for resolving this bound is to use plane waves, i.e. to move to the momentum space via the Fourier transformation:

$$\begin{aligned}
 F(q) &= \int_{-\infty}^{\infty} e^{2\pi i t q} f(t) dt, \\
 f(t) &= \int_{-\infty}^{\infty} e^{-2\pi i t q} F(q) dq.
 \end{aligned} \tag{27}$$

The spectral deformation in the momentum space reads therefore

$$q^2 \leq \frac{\pi^2}{(2\pi)^2} = \frac{1}{4}. \tag{28}$$

Combination of Fourier transforms provides a representation of an identity operator

$$f(t') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i t' q} e^{2\pi i t q} f(t) dt dq. \tag{29}$$

The deformation (28) projects the above identity operator onto

$$\mathbf{P}[f(t')] = \int_{-\infty}^{\infty} \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i t' q} e^{2\pi i t q} dq \right] f(t) dt, \tag{30}$$

Microscopic scaling in the bulk restricts the range of momenta to the interval $-\frac{1}{2} \leq q \leq \frac{1}{2}$. Calculation of the integral in square brackets yields the projection in the position basis, which is the sine kernel

$$K_{\text{Sine}}(t, t') = \frac{\sin(\pi(t' - t))}{\pi(t' - t)}. \tag{31}$$

C Airy kernel. At the soft edge we introduce the scaling variable s as $x/T = r_{\pm} \pm \frac{s}{\sqrt{c}(r_{\pm}N)^{2/3}}$. In the large N and T limit generic bound (15) is transformed into

$$-\frac{d^2}{ds^2} + s \leq 0. \tag{32}$$

On the l.h.s. we recognize the Schrödinger operator with the linear potential. This condition in the position-momentum space (s, q) restricts the range of integration to the parabola $4\pi^2 q^2 + s \leq 0$, which is not well suited for reading out the limiting kernel. To circumvent this problem, Tao introduced a similarity transformation in the momentum space [19]. Alternatively, since we identify the differential Airy operator in (32), we can directly resort to the *Airy transform* [54]

$$F(z) = A[f(t)] = \int_{-\infty}^{\infty} Ai(z - t) f(t) dt \tag{33}$$

and its inverse

$$f(t) = \int_{-\infty}^{\infty} F(z) Ai(z - t) dz. \tag{34}$$

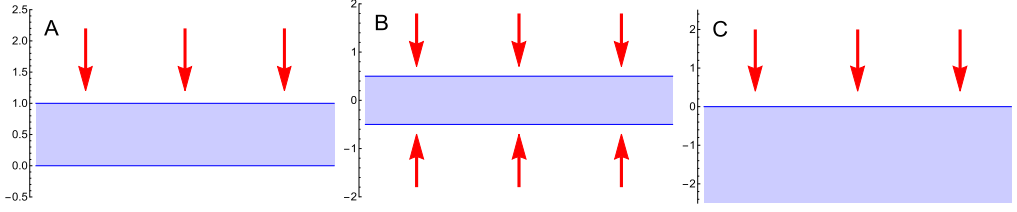


Fig. 2. Regions in the phase space after microscopic scaling at the hard edge (A), in bulk (B) and at the soft edge (C). Red arrows point at the direction of deformation. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Using the Airy transform for the operator bound (32), and the fact that the Airy function fulfills $Ai''(x) = x Ai(x)$ we express the spectral deformation in dual variable t simply as

$$z \leq 0. \quad (35)$$

Combining both Airy transforms we obtain the identity operator

$$f(t') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Ai(t' - z) Ai(t - z) f(t) dt dz \quad (36)$$

The deformation condition (35) turns the above identity operator into a projection

$$\mathbf{P}[f(t')] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^0 Ai(t' - z) Ai(t - z) dz \right] f(t) dt, \quad (37)$$

so the kernel reads

$$K_{Airy}(t, t') = \int_{-\infty}^0 Ai(t' - z) Ai(t - z) dz = \frac{Ai(t') Ai'(t) - Ai'(t') Ai(t)}{t' - t}, \quad (38)$$

where on the r.h.s. we presented the more familiar form of the Airy kernel based on relation

$$\frac{d}{dz} \left[\frac{Ai(t' - z) Ai'(t - z) - Ai'(t' - z) Ai(t - z)}{t' - t} \right] = Ai(t' - z) Ai(t - z). \quad (39)$$

We summarize this section in Fig. 2, by plotting the domain of the projection operator before and after the pertinent microscopic scalings.

3. Bochner theorem and beyond - non-Hermitian Hamiltonians

3.1. Bochner theorem

In a short paper written in 1929 [31] Salomon Bochner has noticed that if an infinite sequence of polynomials $P_n(x)$ satisfies an eigenequation to the second order self-adjoint differential operator

$$p(x) P_n''(x) + q(x) P_n'(x) + r(x) P_n(x) = \lambda_n P_n(x), \quad (40)$$

then $p(x), q(x), r(x)$ must be polynomials of degree 2, 1, and 0, respectively. If additionally polynomials are orthogonal and their support is real, the only solutions are polynomials of Jacobi, Laguerre or Hermite type.

These orthogonal polynomials are associated with classical random matrix ensembles: Gaussian Unitary Ensemble (Hermite), Laguerre Unitary Ensemble (also known as complex Wishart) and Jacobi Unitary Ensemble (null hypothesis in the complex multivariate analysis of variance). They do not lead to any new universality class than what is known for Wishart ensemble. Recently, Bornemann [20], using the spectral projection method, classified the scaling limits of determinantal processes arising from Sturm-Liouville operators.³ They do not lead to any new universality class than what is known for Wishart ensemble. On the other hand, it is known that there are other scaling limits of the kernel in unitary matrix models. These are related to different vanishing of the spectral density at the edge or at the closing gap in the bulk, see [55] for a review. This raises a question whether such limits can be related to spectral projections. To avoid limitations of the Bochner theorem one may look at the class of Hamiltonians with higher powers of momentum operator. Self-adjointness constrains these Hamiltonians to have only even powers of momentum and Krall [56] provided complete classification of orthogonal polynomials to the problem with quartic momenta. However, classification of higher order Bohner-Krall polynomial systems remains still an open problem. While there are some particular examples of sixth [57] and eighth order systems [58], the corresponding weights are only modifications of classical Gaussian, Laguerre and Jacobi weights by Heaviside theta and Dirac delta functions (see [59] for review), which makes them uninteresting from the random matrix theory perspective.

3.2. Non-Hermitian ‘Hamiltonians’

Relaxing the self-adjointness condition admits a broader class of operators. Then one deals with non-Hermitian ‘Hamiltonian’ and two eigenequations to each eigenvalue:

$$\mathcal{H}|P_k\rangle = \lambda_k |P_k\rangle \quad \text{and} \quad \mathcal{H}^\dagger |Q_k\rangle = \lambda_k |Q_k\rangle. \quad (41)$$

Here $|P_k\rangle$ and $\langle Q_k|$ are called left and right eigenfunctions, in the analogy to non-Hermitian matrices. They are no longer orthogonal, but bi-orthogonal

$$\langle Q_k | P_l \rangle = \int Q_k(x) P_l(x) dx = \delta_{kl}. \quad (42)$$

The adjoint Hamiltonian \mathcal{H}^\dagger is defined in a standard way

$$\int f(x) \mathcal{H} g(x) dx = \int (\mathcal{H}^\dagger f(x)) g(x) dx. \quad (43)$$

Now, because of biorthogonality the two sets of eigenfunctions cannot be both polynomials, enlarging the space of possible solutions.

Preiser [39] considered a higher order generalization of Bochner-Krall theorem with restriction that $P_k(x)$ are polynomials in x , while $Q_k(x)$ are polynomials in x^m multiplied by some weight. He found that for the Hamiltonian with third derivative there exists only one such set, which was discovered earlier by Spencer and Fano [38].

Biorthogonal structures appear in multi matrix models, where the correlation kernel is built from biorthogonal functions P_k and Q_k

³ He did not use explicitly Bochner theorem.

$$K_N(x, y) = \sum_{k=0}^{N-1} Q_k(x) P_k(y). \quad (44)$$

Biorthogonality ensures that the kernel is a projection. It is therefore tempting to ask whether such kernels are built of eigenfunctions of a certain ‘Hamiltonian’ and if so, is it possible to obtain the microscopic scaling using spectral projections?

3.3. Singular values of products of complex Gaussian matrices

Let us consider X_k - rectangular matrices of size $(N + \nu_{k-1}) \times (N + \nu_k)$ with complex Gaussian iid entries of zero mean and unit variance. Without loss of generality we assume $\nu_0 = 0$ and $\nu_k > 0$ for $k > 0$. The squared singular values of the product $Y_M = X_1 X_2 \dots X_M$ form a biorthogonal ensemble with the correlation kernel (44). The biorthogonal functions are explicitly given by [42]

$$P_k(x) = G_{1, M+1}^{1,0} \left(\begin{matrix} k+1 \\ 0, -\nu_M, \dots, -\nu_1 \end{matrix} \middle| x \right), \quad (45)$$

$$Q_k(x) = G_{1, M+1}^{M,1} \left(\begin{matrix} -k \\ \nu_M, \dots, \nu_1, 0 \end{matrix} \middle| x \right). \quad (46)$$

Here G stands for the Meijer-G function (see Appendix C). From the differential equation (88) we deduce that polynomials P_k satisfy the eigenproblem ($\mathcal{H}_M P_k = \lambda_k P_k$ with $\lambda_k = k$) of the following differential operator (Hamiltonian)

$$\mathcal{H}_M = x \frac{d}{dx} - \frac{d}{dx} \prod_{j=1}^M \left(x \frac{d}{dx} + \nu_j \right). \quad (47)$$

With the help of the identity $(\frac{d}{dx} x - \nu_j) \frac{d}{dx} = \frac{d}{dx} (x \frac{d}{dx} - \nu_j)$ we immediately obtain its adjoint

$$\mathcal{H}_M^\dagger = -x \frac{d}{dx} - 1 + (-1)^M \frac{d}{dx} \prod_{j=1}^M \left(x \frac{d}{dx} - \nu_j \right). \quad (48)$$

The explicit form (46) and the differential equation (88) prove that Q_k satisfy the eigenequation $\mathcal{H}_M^\dagger Q_k = k Q_k$. Therefore P_k and Q_k are left and right eigenfunctions of a non-Hermitian Hamiltonian.

To probe the microscopic scaling at the edge, we rescale $x = \frac{z}{N}$, which turns the eigenequation for \mathcal{H}_M into

$$\left[\frac{1}{N} z \frac{d}{dz} - \frac{d}{dz} \prod_{j=1}^M \left(z \frac{d}{dz} + \nu_j \right) \right] P_k = \frac{k}{N} P_k. \quad (49)$$

As k is always smaller than N , in the large N limit we look for the functions that satisfy $\langle \psi | \Delta_v^{(M+1)} | \psi \rangle \leq 1$, with

$$\Delta_v^{(M+1)} := -\frac{d}{dz} \prod_{j=1}^M \left(z \frac{d}{dz} + \nu_j \right). \quad (50)$$

In order to continue the analogy to the deformation of the phase-space of Hermitian operators, we have to find the suitable transformation, which will convert the above inequality into an algebraic constraint.

3.4. The Narain transform

In a series of papers [43–45] Narain introduced a broad class of asymmetric transforms, which include many known classical transforms. The Narain transform and its inverse are defined as

$$g(s) = \int_0^\infty k(s, y) f(y) dy, \quad f(y) = \int_0^\infty h(y, s) g(s) ds, \quad (51)$$

where the integral kernels read

$$k(s, y) = 2\gamma x^{\gamma-1/2} G_{p+q, m+n}^{m, p} \left(\begin{matrix} a_1, \dots, a_p, b_1, \dots, b_q \\ c_1, \dots, c_m, d_1, \dots, d_n \end{matrix} \middle| (sy)^{2\gamma} \right), \quad (52)$$

$$h(y, s) = 2\gamma x^{\gamma-1/2} G_{p+q, m+n}^{n, q} \left(\begin{matrix} -b_1, \dots, -b_q, -a_1, \dots, -a_p \\ -d_1, \dots, -d_n, -c_1, \dots, -c_m \end{matrix} \middle| (ys)^{2\gamma} \right). \quad (53)$$

If f has a discontinuity at x , then $\int_0^\infty h(x, s) ds \int_0^\infty k(s, y) f(y) dy$ takes the value $\frac{1}{2}(f(x+0) + f(x-0))$, provided that $\sum a_k + \sum b_k = \sum c_k + \sum d_k$.

3.5. Spectral projection for products of Wishart Matrices

We use the following kernels in the Narain transformation

$$k(s, y) = G_{0, M+1}^{M, 0} \left(\begin{matrix} - \\ v_1, \dots, v_M, 0 \end{matrix} \middle| sy \right), \quad (54)$$

$$h(y, s) = G_{0, M+1}^{1, 0} \left(\begin{matrix} - \\ 0, -v_1, \dots, -v_M \end{matrix} \middle| sy \right). \quad (55)$$

In the space of the dual variable s , the operator $\Delta_v^{(M+1)}$ acts by multiplying by s , as can be easily proven, using identities from Appendix C. The hard edge scaling of the kernel reduces therefore the range of parameter s to $s \leq 1$. Alike in the Hermitian case, the identity operator

$$g(x) = \int_0^\infty \left[\int_0^\infty h(x, s) k(s, y) ds \right] g(y) dy \quad (56)$$

is deformed to

$$\mathbf{P}[f(x)] = \int_0^\infty \left[\int_0^1 h(x, s) k(s, y) ds \right] f(y) dy. \quad (57)$$

We obtain this way the limiting form of the microscopic kernel at the hard edge

$$K_M^{\text{hard}}(x, y) = \int_0^1 G_{0, M+1}^{1, 0} \left(\begin{matrix} - \\ 0, -v_1, \dots, -v_M \end{matrix} \middle| sx \right) G_{0, M+1}^{M, 0} \left(\begin{matrix} - \\ v_1, \dots, v_M, 0 \end{matrix} \middle| sy \right) ds. \quad (58)$$

Note that $G_{0,2}^{1,0}\left(\begin{smallmatrix} - \\ \nu, 0 \end{smallmatrix} \middle| x\right) = x^{\nu/2} J_{\nu}(2\sqrt{x})$ and $G_{0,2}^{1,0}\left(\begin{smallmatrix} - \\ 0, -\nu \end{smallmatrix} \middle| x\right) = x^{-\nu/2} J_{\nu}(2\sqrt{x})$, which yields

$$K_1^{hard}(x, y) = \left(\frac{y}{x}\right)^{\nu/2} \int_0^1 J_{\nu}(2\sqrt{sx}) J_{\nu}(2\sqrt{sy}) ds. \quad (59)$$

This form slightly differs from (24). To understand this discrepancy, let us note that biorthogonal functions can be rescaled as $P_k(x) \rightarrow f(x)P_k(x)$ and $Q_k(x) \rightarrow \frac{1}{f(x)}Q_k(x)$ without altering their biorthogonality. Under such a rescaling kernel is transformed $K(x, y) \rightarrow \frac{1}{f(x)}K(x, y)f(y)$. In our case it is sufficient to take $f(x) = x^{\nu/2}$ and further rescale $(x, y) \rightarrow \frac{1}{4}(x, y)$. The Narain transform can therefore be viewed as a generalization of the Hankel transform at the hard edge.

3.6. Muttalib-Borodin ensemble with the Laguerre weight

As another example we consider the joint pdf of eigenvalues introduced by Muttalib [37] and elaborated later by Borodin [13]

$$P(\lambda_1, \dots, \lambda_n) = C_N \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j| \prod_{1 \leq i < j \leq N} |\lambda_i^{\theta} - \lambda_j^{\theta}| \prod_{k=1}^N \lambda_k^{\alpha} e^{-\lambda_k} d\lambda_k, \quad (60)$$

with $\alpha > -1$, and $\theta \geq 0$. Eigenvalues form a determinantal point process with a correlation kernel given by the bi-orthogonal functions (44). Here P_k is a polynomial of order k , while Q_k is a polynomial in x^{θ} multiplied by the Laguerre weight. For integer values of θ Konhauser provides the explicit form of Q [40, eq. (5)]

$$Q_k(x) = x^{\alpha} e^{-x} \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{x^{j\theta}}{\Gamma(j\theta + \alpha + 1)}, \quad (61)$$

while Carlitz gives the explicit form of polynomials [60, eq. (9)]

$$P_k(x) = \frac{1}{k!} \sum_{i=0}^k \frac{x^i}{i!} \sum_{j=0}^i (-1)^j \binom{i}{j} \frac{\Gamma(k + \frac{j+\alpha+1}{\theta})}{\Gamma(k)}. \quad (62)$$

For $\theta = 1$ this reduces to the Laguerre orthogonal polynomials, while the case $\theta = 2$ was considered by Preiser [39] in an attempt to extend Bohner-Krall theorem. Polynomials satisfy the eigenvalue equation $\mathcal{H}P_k = \lambda_k P_k$, with $\lambda_k = \theta k$ of the following differential operator [40]

$$\mathcal{H} = \left(\frac{d}{dx} x + \alpha - x \right) \left[\left(1 - \frac{d}{dx} \right)^{\theta} - 1 \right]. \quad (63)$$

Konhauser showed also that $Z_k = x^{-\alpha} e^x Q_k(x)$, a polynomial in x^{θ} , satisfies [40, eq. (10)]

$$\left(\frac{d}{dx} \right)^{\theta} x^{\alpha+1} \frac{d}{dx} Z_k - x^{\alpha+1} \frac{d}{dx} Z_k = -x^{\alpha} \theta k Z_k. \quad (64)$$

Then it is easy to show that Q_k satisfies the eigenequation $\mathcal{H}^{\dagger} Q_k = \lambda_k Q_k$ to the same eigenvalues as P_k . The differential operator

$$\mathcal{H}^\dagger = \left[1 - \left(1 + \frac{d}{dx} \right)^\theta \right] \left(-\alpha + x + x \frac{d}{dx} \right) \quad (65)$$

is the adjoint of \mathcal{H} . We probe the hard edge by introducing a new variable $x = uN^{-\frac{1}{\theta}}$. In the large N limit, having in mind that $k < N$, from the eigenequation for \mathcal{H}^\dagger we obtain

$$-\frac{1}{\theta} \left(\frac{d}{du} \right)^\theta \left(u \frac{d}{du} - \alpha \right) \leq 1. \quad (66)$$

A change of variable $u = \theta z^{1/\theta}$ turns this condition into a more familiar form

$$-\frac{d}{dz} \prod_{j=1}^{\theta} \left(z \frac{d}{dz} + v_j \right) \leq 1, \quad (67)$$

with

$$v = -\frac{1}{\theta}, -\frac{2}{\theta}, -\frac{3}{\theta}, \dots, -\frac{\theta-1}{\theta}, -\frac{\alpha}{\theta}. \quad (68)$$

We now take

$$k(s, y) = G_{0, \theta+1}^{\theta, 0} \left(0, -\frac{1}{\theta}, \dots, -\frac{\theta-1}{\theta}, -\frac{\alpha}{\theta} \middle| sy \right), \quad (69)$$

$$h(y, s) = G_{0, \theta+1}^{1, 0} \left(\frac{\alpha}{\theta}, 0, \frac{1}{\theta}, \dots, \frac{\theta-1}{\theta} \middle| sy \right). \quad (70)$$

Again, using the identities from Appendix C one can show the identity $\int_0^\infty k(s, z) (\mathcal{H}^\dagger f(z)) dz = \int_0^\infty s k(s, z) f(z) dz$. This means that the condition (67) in the dual space is equivalent to $s \leq 1$. This allows us to read out the form of the kernel

$$K(y, x) = \int_0^1 G_{0, \theta+1}^{1, 0} \left(\frac{\alpha}{\theta}, 0, \frac{1}{\theta}, \dots, \frac{\theta-1}{\theta} \middle| sx \right) G_{0, \theta+1}^{\theta, 0} \left(0, -\frac{1}{\theta}, \dots, -\frac{\theta-1}{\theta}, -\frac{\alpha}{\theta} \middle| sy \right) ds. \quad (71)$$

Note also that the truncation condition $s \leq 1$ was obtained from the consideration of \mathcal{H}^\dagger , therefore the kernel has now interchanged arguments. Using (87) we also write an equivalent kernel

$$\left(\frac{y}{x} \right)^{\frac{\alpha}{\theta}} K(y, x) = \int_0^1 G_{0, \theta+1}^{1, 0} \left(0, -\frac{\alpha}{\theta}, -\frac{\alpha-1}{\theta}, \dots, -\frac{\alpha-\theta+1}{\theta} \middle| sx \right) G_{0, \theta+1}^{\theta, 0} \left(\frac{\alpha}{\theta}, \frac{\alpha-1}{\theta}, \dots, \frac{\alpha-\theta+1}{\theta}, 0 \middle| sy \right) ds, \quad (72)$$

which corresponds to the form obtained by Kuijlaars and Stivigny [61, Theorem 5.1].

4. Summary

We explained the spectral projection method on the example of Wishart ensemble, rederiving three classical microscopic universalities. We also linked the Marchenko-Pastur distribution with

the Bohr-Sommerfeld quantization condition of the hydrogen atom. Such a link is intuitively expected, because the Dyson electrostatic analogy in the limit of large matrices allows one to solve random matrix models using the saddle point approximation - the same mathematical method which gives the WKB approximation in quantum mechanics, with the correspondence $\frac{1}{\hbar} \leftrightarrow N$. To the best of our knowledge such an interpretation was not emphasized in the literature.

Bochner theorem provides limitations on the applications of spectral projection method to orthogonal polynomials. We demonstrated that the spectral projection method can be extended to biorthogonal systems. To demonstrate the power of this approach, we recalculated the hard edge Meijer-G kernel. In the paper [42], where all correlations functions for products of Wishart matrices were found for the first time, the problem of calculating the microscopic limit at the hard edge was not tackled, despite that such a result was expected. This task was performed several months later as a subject of a separate work [62]. This impressive calculation required refined integral representations and careful saddle point analysis. In this work we show how the calculations can be reduced to just a few lines. Moreover, to the best of our knowledge, the Narain transform was not used before in random matrix theory.

Rapid progress in random matrix theory in last three decades has brought plethora of new microscopic universality classes. Despite of many examples of microscopic universalities, there is lack of their systematic classification. The spectral projection method adopted to non-Hermitian Hamiltonians and possible generalizations of Bochner theorem for higher order differential operators⁴ offer a new perspective on this problem. Certainly, this program is a challenging mathematical problem, which we do not attempt to solve in this work.

This work raises a series of fundamental questions related to possible generalization of Bochner theorem in the context of random matrix theory. Is it possible to reframe all universality classes in this language? Will this classification be predictive for constructing new types of random matrix models? Can one infer the microscopic kernels of non-Hermitian ensembles from a ‘complex version’ of Bochner theorem? Can one mimic this construction for pfaffian processes for $\beta = 1, 4$? If so, is there a link to Quantum Hall Effect [64]? We leave these questions open but we think that the presented method has also pedagogical value. It offers an easy and intuitive way to recover not only the classical universality classes, but also more involving Meijer-G functions. Combining physical intuition with mathematics may provide in such a way new insights even in standard problems.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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⁴ After completion of the paper, Oleg Evnin has pointed to us a recent paper [63], proposing the classification of cubic extension of Bochner theorem. This construction, however, does not refer to random matrix theory.

Appendix A. Mapping Wishart onto 2D hydrogen atom

The time-independent Schrödinger equation in 2D with the potential $V(r) = -Ze^2/r$ in the cylindrical coordinates reads

$$\left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) - Ze^2/r \right] \phi(r, \varphi) = E\phi(r, \varphi). \quad (73)$$

An Ansatz $\phi(r, \varphi) = R(r)e^{il\varphi}/\sqrt{2\pi}$ separates variables. Setting the physical constants $Ze^2 = 1$, $2m = 1$, $\hbar = 1$ and changing variables as $\rho = \lambda r$, $E = -1/4\lambda^2$ we arrive at the equation for the radial part

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + \frac{\lambda}{\rho} - \frac{l^2}{\rho^2} - \frac{1}{4} \right) R(\rho) = 0. \quad (74)$$

Upon identification $2l = |\alpha|$ and $2\lambda = 1 + 2k + \alpha$ we obtain the equation (6) for the function building the kernel.

Appendix B. WKB analysis of the macroscopic spectral density

The spectral density is calculated from the kernel as

$$\rho(x) = \frac{1}{N} K(x, x) = \frac{1}{N} \sum_{k=0}^{N-1} \psi_k^2(x). \quad (75)$$

In the large N limit the sum can be approximated by an integral over the variable $t = k/N$

$$\rho(x) \xrightarrow{N \rightarrow \infty} \int_0^1 \psi_t^2(x) dt. \quad (76)$$

Taking the equation (6) for ψ_k , rescaling $x \rightarrow Tx$ and setting $t = k/N$, we obtain

$$\frac{1}{T^2} \left(\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} \right) \psi_t(x) = \left(\frac{1}{4} + \frac{(1-c)^2}{4x^2} - \frac{ct}{x} - \frac{1-c}{2x} \right) \psi_t(x) \equiv (V(x) - E) \psi_t(x). \quad (77)$$

We also note that up to a term $1/4x^2$, which is irrelevant in the asymptotic analysis, the operator on the lhs of (77) is minus square of the radial momentum $p_r(x) = -i\hbar \left(\frac{1}{r} + \frac{d}{dr} \right)$. Using the WKB Ansatz $\psi(x) = A(x)e^{T\phi(x)}$, we obtain the general solution

$$\psi_t(x) = \frac{1}{\sqrt{x p_r(x)}} \left(C_+ e^{iT \int^x p_r(x') dx'} + C_- e^{-iT \int^x p_r(x') dx'} \right). \quad (78)$$

Matching condition at each of the turning points gives two forms of the solution

$$\psi(x) = \frac{C}{\sqrt{x p_r(x)}} \cos \left[-\frac{\pi}{4} + T \int_{x_-}^x dx' p_r(x') \right] = \frac{C'}{\sqrt{x p_r(x)}} \cos \left[-\frac{\pi}{4} + T \int_x^{x_+} dx' p_r(x') \right]. \quad (79)$$

Uniqueness of the solution irrespective of the choice of turning point leads to the quantization condition

$$T \oint p_r(x) dx = 2\pi \left(n + \frac{1}{2} \right), \quad n \in \mathbb{N}. \quad (80)$$

Note that for the calculation of the spectral density, ψ_t^2 is needed. For large T it is a rapidly oscillating function and the oscillations average out and only the average of \cos^2 , which is $1/2$, is relevant⁵

$$\psi_t^2(x) = \begin{cases} 0 & \text{for } x < x_- \text{ or } x > x_+ \\ \frac{C}{2xp(x,t)} & \text{for } x_- < x < x_+ \end{cases}. \quad (81)$$

The turning points are

$$x_{\pm}(t) = 1 - c + 2ct \pm 2\sqrt{ct(1 + ct - c)}. \quad (82)$$

The spectral density is therefore given by

$$\rho(x) = \int_0^1 dt \frac{C}{\sqrt{2c(1 + 2tx - x) - c^2 - (x - 1)^2}} \chi_{x_- < x < x_+} = \frac{C}{2cx} \sqrt{(x - (1 - \sqrt{c})^2)((1 + \sqrt{c})^2 - x)}, \quad (83)$$

where χ_A is equal to 1 when A is true and 0 for A false. Setting $C = \frac{1}{\pi}$ normalizes the density.

Appendix C. Some properties of Meijer-G functions

The Meijer-G functions are defined as an integral

$$G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds, \quad (84)$$

where $\Gamma(z)$ is the Euler gamma function. The integration contour L is chosen to separate all poles of $\prod_{j=1}^m \Gamma(b_j - s)$ from the poles of $\prod_{j=1}^n \Gamma(1 - a_j + s)$ (see also [65], §5.2 for details). By definition, they are symmetric in its first m and last $q - m$ lower parameters. When first and the last lower parameter differ by an integer number, they can be interchanged

$$G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, b_2, \dots, b_{q-1}, b_q \end{matrix} \middle| z \right) = (-1)^{b_q - b_1} G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_q, b_2, \dots, b_{q-1}, b_1 \end{matrix} \middle| z \right). \quad (85)$$

The following differential operator acts by increasing first lower indices

$$\left(-z \frac{d}{dz} + b_1 \right) G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z \right) = G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1 + 1, b_2, \dots, b_q \end{matrix} \middle| z \right). \quad (86)$$

Combining this with (85), we obtain the operator $z \frac{d}{dz} - b_q$, which increases last lower indices. Multiplication by the argument allows one to increase all indices

⁵ This can be rephrased more rigorously in terms of weak convergence.

$$z^\alpha G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = G_{p,q}^{m,n} \left(\begin{matrix} a_1 + \alpha, \dots, a_p + \alpha \\ b_1 + \alpha, \dots, b_q + \alpha \end{matrix} \middle| z \right). \quad (87)$$

Meijer-G functions satisfy the following differential equation

$$\left[(-1)^{p-m-n} z \prod_{j=1}^p \left(z \frac{d}{dz} - a_j + 1 \right) - \prod_{j=1}^q \left(z \frac{d}{dz} - b_j \right) \right] G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = 0. \quad (88)$$

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